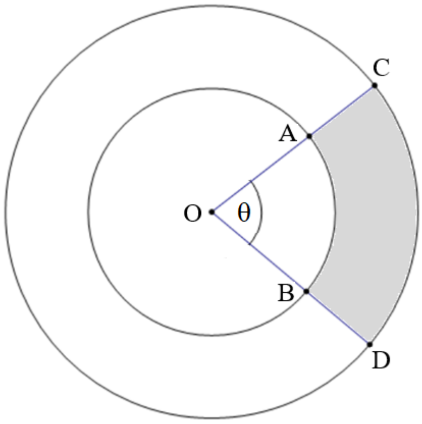
Mathematics: analysis and approaches
Higher Level
Paper 2 (set C)

Markscheme v1

1. [Maximum mark: 5]

The diagram below shows two circles which have the same centre O. The smaller circle has a radius of 12 cm and the larger circle has a radius of 20 cm. The two arcs AB and CD have the same central angle θ, where $\theta = 1.3$ radians.
Find the area of the shaded region.



Markscheme:

area of shaded region = (area of sector COD) – (area of sector AOB) **R1**

Use of the formula $A = \frac{1}{2}r^2\theta$ **(M1)**

area of sector COD $= 260\text{ cm}^2$ **A1**

area of sector AOB $= 93.6\text{ cm}^2$ **A1**

area of shaded region $\approx 166\text{ cm}^2$ **A1**

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2. [Maximum mark: 5]

Two lines have the vector equations $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$.

Find the obtuse angle between the lines.

Markscheme:

Recognising $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 3 \\ 1 \\ -2 \end{pmatrix}$ as the directional vectors of the lines **(A1)**

Use of the formula $\cos\theta = \frac{v_1w_1 + v_2w_2 + v_3w_3}{|\mathbf{v}||\mathbf{w}|}$ **(M1)**

$\cos\theta = \frac{7}{\sqrt{6}\sqrt{14}}$ **A1**

$\theta \approx 40.203...°$ **(A1)**

Obtuse angle $\approx 180° - 40.203...° \approx 139.797...° \Rightarrow \theta \approx 140°$ **A1**

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3. [Maximum mark: 5]

Find the coefficient of the x^3 term in the expansion of $\left(\frac{2}{3}x+3\right)^8$.

Markscheme:

Recognising general term of expansion as $\binom{8}{r}\left(\frac{2}{3}x\right)^{8-r} 3^r$ **M1A1**

considering exponent of x: $8-r=3 \Rightarrow r=5$ **A1**

Attempting to solve coefficient of general term for $r=5$ **(M1)**

coefficient of x^3 term $=4032$ **A1**

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4. [Maximum mark: 6]

The table below shows the marks earned on a quiz by a group of students.

Mark	1	2	3	4	5
Number of students	8	7	c	9	1

The median is 3 and the mode is 4 for the set of marks. Find the **three** possible values of c.

Markscheme:

Since 4 is the mode, then $c<9$ **R1**

The total number of students is $8+7+c+9+1=c+25$ **A1**

Since 3 is the median, then $\frac{c+25}{2}>8+7$ **R1**

$\frac{c+25}{2}>8+7 \Rightarrow c+25>30 \Rightarrow c>5$ **A1**

Thus, the three possible values of c are 6, 7, 8 **A2**

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5. [Maximum mark: 6]

Consider the complex number $z = \frac{\sqrt{2}}{1-\text{i}} - \text{i}$.

(a) Show that z can be expressed, in the form $x + y\text{i}$, as $z = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}-2}{2}\right)\text{i}$. [2]

(b) (i) Find the **exact** value of the modulus of z.

(ii) Find the argument θ of z, where $-\pi < \theta \le \pi$. [4]

Markscheme:

(a) Using the complex conjugate to solve for z **M1**

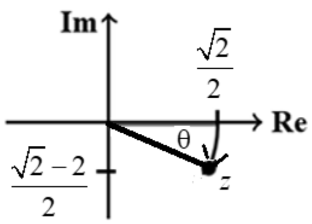
$$z = \frac{\sqrt{2}}{1-\text{i}} \cdot \frac{1+\text{i}}{1+\text{i}} - \text{i} = \frac{\sqrt{2}+\text{i}\sqrt{2}}{1-\text{i}^2} - \text{i} = \frac{\sqrt{2}+\text{i}\sqrt{2}}{2} - \text{i} = \frac{\sqrt{2}+\text{i}\sqrt{2}-2\text{i}}{2}$$ **A1**

Thus, $z = \frac{\sqrt{2}}{2} + \left(\frac{\sqrt{2}-2}{2}\right)\text{i}$ **AG**

(b) (i) Recognising that modulus $= |z| = \sqrt{x^2 + y^2}$ **(M1)**

$|z| = \sqrt{2-\sqrt{2}}$ **A1**

(ii) Recognising that, as z is in the 4th quadrant, argument $\arg(z) = \theta = \tan^{-1}\left(\frac{y}{x}\right)$ **(M1)**



$$\arg(z) = \theta = \tan^{-1}\left(\frac{\frac{\sqrt{2}-2}{2}}{\frac{\sqrt{2}}{2}}\right) = \tan^{-1}\left(\frac{\sqrt{2}-2}{\sqrt{2}}\right) = -0.392699... \Rightarrow \theta \approx -0.393$$ **A1**

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6. [Maximum mark: 7]

(a) Express $\frac{1}{2x^2+7x-4}$ in partial fractions; i.e. as the sum of two fractions. [4]

(b) Given that $\int_1^4 \frac{9}{2x^2+7x-4}\,\text{d}x = \ln k$, find the **exact** value of k. [3]

Markscheme:

(a) $\dfrac{1}{2x^2+7x-4}=\dfrac{1}{(2x-1)(x+4)}=\dfrac{A}{2x-1}+\dfrac{B}{x+4}$ **M1**

multiply both sides by $(2x-1)(x+4)$, to give $1=A(x+4)+B(2x-1)$ **A1**

Using appropriate values of x (e.g. $x=\dfrac{1}{2}$, $x=-4$) to find $A=\dfrac{2}{9}$ and $B=-\dfrac{1}{9}$ **A1**

thus, $\dfrac{1}{2x^2+7x-4}=\dfrac{\frac{2}{9}}{2x-1}-\dfrac{\frac{1}{9}}{x+4}$ or $\dfrac{1}{2x^2+7x-4}=\dfrac{2}{9(2x-1)}-\dfrac{1}{9(x+4)}$ **A1**

(b) Using result in part (a) to show $\displaystyle\int_1^4 \frac{9}{2x^2+7x-4}\,dx=\int_1^4\left(\frac{2}{2x-1}-\frac{1}{x+4}\right)dx$ **M1**

$\displaystyle\int_1^4\left(\frac{2}{2x-1}-\frac{1}{x+4}\right)dx=\Big[\ln|2x-1|-\ln|x+4|\Big]_1^4$ **A1**

$\Big[\ln|2x-1|-\ln|x+4|\Big]_1^4=\ln\dfrac{35}{8}$, thus $k=\dfrac{35}{8}$ **A1**

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7. [Maximum mark: 6]

(a) Write down the Maclaurin expansion of e^x up to the term in x^4. [1]

(b) Find the Maclaurin expansion of e^{x^2} up to the term in x^4. [2]

(c) Hence, find the Maclaurin expansion of e^{x+x^2} up to the term in x^4. [3]

Markscheme:

(a) $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}$ or $e^x=1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}$ **A1**

(b) Substituting $x=x^2$ into the Maclaurin series for e^x **M1**

$e^{x^2}=1+x^2+\dfrac{(x^2)^2}{2}+\dfrac{(x^2)^3}{6}+\dfrac{(x^2)^4}{24}=1+x^2+\dfrac{x^4}{2}+\dfrac{x^6}{6}+\dfrac{x^8}{24}$ **A1**

(c) $e^{x+x^2}=e^x\cdot e^{x^2}=\left(1+x^2+\dfrac{x^4}{2}+\dfrac{x^6}{6}+\dfrac{x^8}{24}\right)\left(1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}\right)$ **A1**

Expanding out brackets, considering only terms up to and including x^4:

$e^{x+x^2}=\left(1+x^2+\dfrac{x^4}{2}\right)\left(1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}\right)=1+x+\dfrac{x^2}{2}+\dfrac{x^3}{6}+\dfrac{x^4}{24}+x^2+x^3+\dfrac{x^4}{2}+\ldots+\dfrac{x^4}{2}+\ldots$ **A1**

$e^{x+x^2}=1+x+\dfrac{3}{2}x^2+\dfrac{7}{6}x^3+\dfrac{25}{24}x^4$ **A1**

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8. [Maximum mark: 6]

Consider the following system of equations

$$\begin{aligned} 2x + y + 6z &= 0 \\ 4x + 3y + 14z &= 4 \\ 2x - 2y + (\alpha - 2)z &= \beta - 12 \end{aligned}$$

Find the conditions on α and β for which

(a) the system has no solutions; [2]

(b) the system has only one solution; [2]

(c) the system has an infinite number of solutions. [2]

Markscheme:

using row operations

$$\left(\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 4 & 3 & 14 & 4 \\ 2 & -2 & \alpha-2 & \beta-12 \end{array}\right) \underset{\Rightarrow}{(R1 - R3 \to R3)} \left(\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 4 & 3 & 14 & 4 \\ 0 & -3 & \alpha-8 & \beta-12 \end{array}\right)$$

$$\underset{\Rightarrow}{(R2 - 2R1 \to R2)} \left(\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & -3 & \alpha-8 & \beta-12 \end{array}\right)$$

$$\underset{\Rightarrow}{(R3 - 3R2 \to R3)} \left(\begin{array}{ccc|c} 2 & 1 & 6 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \alpha-2 & \beta \end{array}\right)$$

(a) no solutions when $\alpha = 2$ and $\beta \neq 0$ **A2**

(b) one solution when $\alpha \neq 2$ **A2**

(c) infinite solutions when $\alpha = 2$ and $\beta = 0$ **A2**

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9. [Maximum mark: 7]

Consider the differential equation $x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \frac{1}{x}$, $x > 0$ such that $y = 1$ when $x = 1$. Show that the solution to this differential equation is $y = \frac{x^2 + 1}{2x^3}$.

Markscheme:

Rearranging differential equation: $x\frac{\mathrm{d}y}{\mathrm{d}x} + 3y = \frac{1}{x} \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{3}{x}y = \frac{1}{x^2}$ **M1**

Calculating integrating factor: $\mathrm{IF} = \mathrm{e}^{\int \frac{3}{x}\mathrm{d}x} = \mathrm{e}^{3\ln x} = x^3$ **M1A1**

$$x^3\left(\frac{dy}{dx}+\frac{3}{x}y\right)=x^3\left(\frac{1}{x^2}\right) \Rightarrow \int x^3\left(\frac{dy}{dx}+\frac{3}{x}y\right)dx=\int x\,dx$$ **M1**

$$\int x^3\left(\frac{dy}{dx}+\frac{3}{x}y\right)dx=\int x\,dx \Rightarrow x^3y=\frac{1}{2}x^2+C$$ **A1**

Attempting to solve for solution with given conditions of x and y **M1**

$$x=1 \text{ when } y=1:\ (1)^3(1)=\frac{1}{2}(1)^2+C \Rightarrow C=\frac{1}{2}$$ **A1**

$$x^3y=\frac{1}{2}x^2+\frac{1}{2}=\frac{x^2+1}{2} \Rightarrow y=\frac{x^2+1}{2x^3} \quad \textbf{\textit{QED}}$$ **AG**

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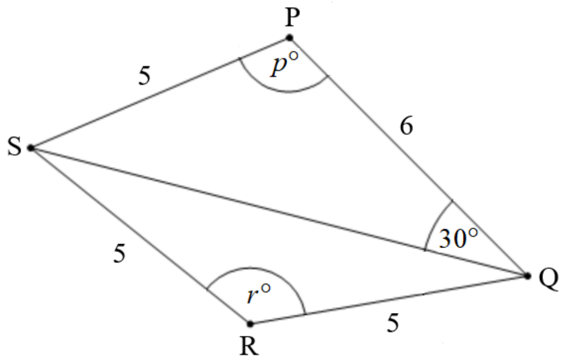
10. [Maximum mark: 15]

The diagram shows the quadrilateral PQRS.
Angle QPS and angle QRS are obtuse.

$PQ=6\text{cm},\ QR=5\text{cm},\ RS=5\text{cm},\ PS=5\text{cm},$
$P\hat{Q}S=30°,\ Q\hat{P}S=p°,\ Q\hat{R}S=r°$



(a) Use the sine rule to show that $QS=10\sin p$. [1]

(b) Use the cosine rule in triangle PQS to find another expression for QS. [3]

(c) (i) Hence, find p, giving your answer to two decimal places.

(ii) Find QS. [6]

(d) (i) Find r.

(ii) Hence, or otherwise, find the area of triangle QRS. [5]

<u>Markscheme</u>:

(a) Using Sine rule: $\frac{\sin 30°}{5}=\frac{\sin p}{QS}$ **M1**

$QS=10\sin p$ ***QED*** **AG**

(b) Attempting to use the Cosine rule **(M1)**

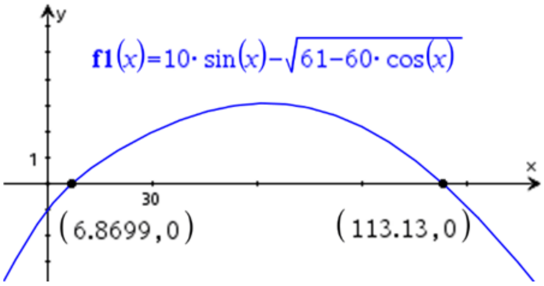
$QS^2=5^2+6^2-2(5)(6)\cos p$ **A1**

$QS=\sqrt{61-60\cos p}$ **A1**

(c) (i) Attempting to solve the equation $10\sin p = \sqrt{61-60\cos p}$ **M1**

Recognising that $90 < p < 180$ **R1**

Solving equation by graphing or with GDC solver **A1**



thus, $p \approx 113.13$ **A1**

(c) (ii) Substituting into equation $\text{QS} = \sqrt{61-60\cos p}$ **M1**

$\text{QS} \approx 9.20\,\text{cm}$ **A1**

(d) (i) Attempting to use the Cosine rule **(M1)**

$(9.19615...)^2 = 5^2 + 5^2 - 2(5)(5)\cos r$ **A1**

$r \approx 134$ **A1**

(ii) Substituting r into area of triangle formula **M1**

Area $\approx 9.03\ \text{cm}^2$ **A1**

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11. [Maximum mark: 21]

A continuous random variable X has probability density function f defined by

$$f(x) = \begin{cases} \frac{\pi}{3}\sin\left(\frac{\pi}{2}x\right), & 0 \le x \le 1 \\ mx + b, & 1 \le x \le k \\ 0, & \text{otherwise} \end{cases}$$

(a) Given that f is continuous on the interval $0 \le x \le k$ and that the graph of f intersects the x-axis at $(k, 0)$, show that $k = \frac{\pi + 2}{\pi}$. [5]

(b) Find the value of m and the value of b. [3]

(c) Sketch the graph of $y = f(x)$. [2]

(d) Write down the mode of X. [1]

(e) Given that $\int_1^{\frac{\pi+2}{\pi}}\left[x(mx+b)\right]dx=\frac{3\pi+2}{9\pi}$, find the **exact** value of the mean of X. [7]

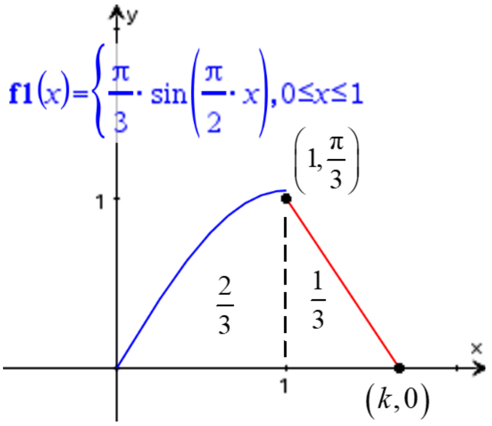
(f) Find the value of the median of X. [3]

Markscheme:

(a) Attempting to integrate $\frac{\pi}{3}\int_0^1\sin\left(\frac{\pi}{2}x\right)dx$ **M1**

$\frac{\pi}{3}\int_0^1\sin\left(\frac{\pi}{2}x\right)dx=\frac{2}{3}$ **A1**

Recognising that the line segment (on $y=mx+b$) must start at $\left(1,\frac{\pi}{3}\right)$ and end at $(k,0)$ **R1**



Recognising that, since total area under p.d.f. is one, the area of triangular region must be $\frac{1}{3}$ **R1**

$\frac{1}{2}\cdot\frac{\pi}{3}(k-1)=\frac{1}{3}$ **A1**

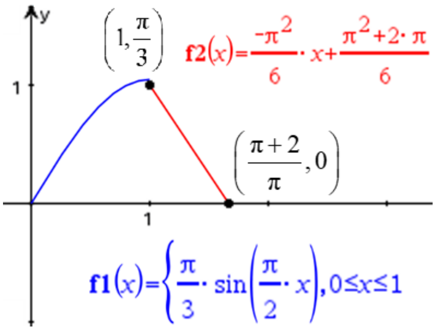
$k=\frac{\pi+2}{\pi}$ ***QED*** **AG**

(b) $m=-\frac{\pi^2}{6}$ **A1**

Substituting into equation for a straight line **M1**

$b=\frac{\pi^2+2\pi}{6}$ **A1**

(c)



A2

(d) Mode is $x=1$ **A1**

(e) Recognising that $\mathrm{E}(X)=\frac{\pi}{3}\int_0^1\left[x\sin\left(\frac{\pi}{2}x\right)\right]\mathrm{d}x+\int_1^{\frac{\pi+2}{\pi}}\left[x\left(-\frac{\pi^2}{6}x+\frac{\pi^2+2\pi}{6}\right)\right]\mathrm{d}x$ **M1**

$$\mathrm{E}(X)=\frac{\pi}{3}\int_0^1\left[x\sin\left(\frac{\pi}{2}x\right)\right]+\frac{3\pi+2}{9\pi}$$ **A1**

Attempting to use Integration by parts **(M1)**

$$\int x\sin\left(\frac{\pi}{2}x\right)\mathrm{d}x=-\frac{2x}{\pi}\cos\left(\frac{\pi}{2}x\right)+\frac{2}{\pi}\int\cos\left(\frac{\pi}{2}x\right)\mathrm{d}x$$ **A1**

$$\int x\sin\left(\frac{\pi}{2}x\right)\mathrm{d}x=-\frac{2x}{\pi}\cos\left(\frac{\pi}{2}x\right)+\frac{4}{\pi^2}\sin\left(\frac{\pi}{2}x\right)$$ **A1**

$$\mathrm{E}(X)=\frac{\pi}{3}\int_0^1 x\sin\left(\frac{\pi}{2}x\right)\mathrm{d}x+\frac{3\pi+2}{9\pi}$$ **A1**

$$\mathrm{E}(X)=\frac{3\pi+14}{9\pi}$$ **A1**

(f) Let m be the median.

Since $\frac{\pi}{3}\int_0^1\sin\left(\frac{\pi}{2}x\right)\mathrm{d}x=\frac{2}{3}$, then $0<m<1$ **R1**

Attempting to solve for m in the equation $\frac{\pi}{3}\int_0^m\sin\left(\frac{\pi}{2}x\right)\mathrm{d}x=\frac{1}{2}$. **(M1)**

$$\text{nSolve}\left(\frac{\pi}{3}\cdot\int_0^m\sin\left(\frac{\pi}{2}\cdot x\right)\mathrm{d}x=\frac{1}{2},m\right)\quad 0.83913875349$$

$m\approx 0.839$ **A1**

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12. [Maximum mark: 21]

The function g is defined as $g(x) = \mathrm{e}^x + \dfrac{1}{2\mathrm{e}^x}$, $x \in \mathbb{R}$.

(a) (i) Explain why the inverse function g^{-1} does not exist.

(ii) The line L intersects the curve $y = g(x)$ at points A and B where $x = -1$ at A and $x = 1$ at B. Show that the equation of L is $y = \dfrac{\mathrm{e}^2 - 1}{4\mathrm{e}}x + \dfrac{3\mathrm{e}^2 + 3}{4\mathrm{e}}$.

(iii) Point C is on the curve $y = g(x)$. The line tangent to the curve $y = g(x)$ at C is parallel to L. Find the coordinates of C. [13]

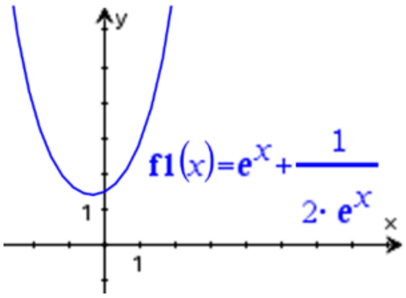
(b) The domain of g is now restricted to $x \geq 0$.

(i) Find an expression for $g^{-1}(x)$.

(ii) Find the volume generated when the region bounded by the curve $y = g(x)$ and the lines $x = 0$ and $y = 4$ is rotated through an angle of 2π radians about the y-axis. [8]

Markscheme:

(a) (i) Recognising that g fails the horizontal line test because a horizontal line can intersect the graph at more than one point; hence, g^{-1} does not exist **R2**



(ii) $g(1) = \dfrac{2\mathrm{e}^2 + 1}{2\mathrm{e}}$, therefore A has coordinates $\mathrm{A}\left(1, \dfrac{2\mathrm{e}^2 + 1}{2\mathrm{e}}\right)$ **A1**

$g(-1) = \dfrac{2 + \mathrm{e}^2}{2\mathrm{e}}$, therefore B has coordinates $\mathrm{B}\left(-1, \dfrac{2 + \mathrm{e}^2}{2\mathrm{e}}\right)$ **A1**

gradient of L: $m = \dfrac{\mathrm{e}^2 - 1}{4\mathrm{e}}$ **A1**

Substituting into the equation for a straight line **(M1)**

equation of L: $y - \dfrac{2\mathrm{e}^2 + 1}{2\mathrm{e}} = \dfrac{\mathrm{e}^2 - 1}{4\mathrm{e}}(x - 1)$ **A1**

$y = \dfrac{\mathrm{e}^2 - 1}{4\mathrm{e}}x - \dfrac{3\mathrm{e}^2 + 3}{4\mathrm{e}}$ ***QED*** **AG**

(iii) Attempting to find $g'(x)$ **M1**

$g'(x) = e^x - \dfrac{1}{2e^x}$ **A1**

Attempting to solve the equation $e^x - \dfrac{1}{2e^x} = \dfrac{e^2 - 1}{4e}$ **(M1)**

$x \approx 0.057811016577...$ **A1**

$g(0.057811016577...) \approx 1.53142889531...$ **A1**

coordinates of C are approximately $(0.0578, 1.53)$ **A1**

(b) (i) $y = e^x + \dfrac{1}{2e^x} \;\Rightarrow\; x = e^y + \dfrac{1}{2e^y}$ **M1**

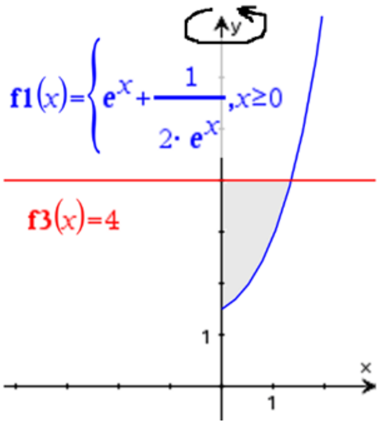
$x = e^y + \dfrac{1}{2e^y} \;\Rightarrow\; 2(e^y)^2 - 2xe^y + 1 = 0$ **A1**

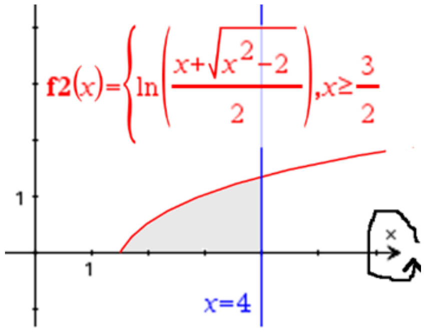
$e^y = \dfrac{x \pm \sqrt{x^2 - 2}}{2} \;\Rightarrow\; y = \ln\left(\dfrac{x \pm \sqrt{x^2 - 2}}{2}\right)$ **A1**

Recognising that, since domain of $g(x)$ is $x \geq 0$, domain of $g^{-1}(x)$ is $x \geq g(0) \;\Rightarrow\; x \geq \dfrac{3}{2}$ **A1**

thus, $g^{-1}(x) = \ln\left(\dfrac{x + \sqrt{x^2 - 2}}{2}\right)$ where $x \geq \dfrac{3}{2}$ **A1**

(ii) Recognising that the volume of the solid formed by rotating the region bounded by $g(x)$ about the *y*-axis is equal to the volume of the solid formed by rotating the region bounded by $g^{-1}(x)$ about the *x*-axis **R1**





volume $= \pi \displaystyle\int_{3/2}^{4} \left[\ln\left(\dfrac{x + \sqrt{x^2 - 2}}{2}\right)\right]^2 dx$ **M1**

$$\pi \cdot \int_{\frac{3}{2}}^{4} \left(\ln\left(\frac{x+\sqrt{x^2-2}}{2}\right)\right)^2 dx \qquad 6.9086056902$$

volume ≈ 6.91 units^3 **A1**